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Analytical Solution for Space - Time Fractional Telegraph Equation via Laplace Transform

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Abstract: *In this paper, a numerical algorithm based on new homotopy perturbation transform method (HPTM) is proposed to obtain space and time fractional telegraph equations. The fractional derivatives are taken in the Caputo sense. The new homotopy perturbation transform method is combined form of Laplace transform, homotopy perturbation method and He's polynomials. The results obtained by the proposed technique show that the approach is easy to implement and computationally very attractive.*

Keywords: Laplace transform; Homotopy perturbation method; He's polynomials; fractionaltelegraph equations

I. Introduction

Fractional differential equations have gained importance and popularity, mainly due to its demonstrated applications in science and engineering. For example, these equations are increasingly used to model problems in research areas as diverse as dynamical systems, mechanical systems, control, chaos, chaos synchronization, continuous-time random walks, anomalous diffusive and subdiffusive systems, unification of diffusion and wave propagation phenomenon and others. The most important advantage of using fractional differential equations in these and other applications is their non-local property. It is well known that the integer order differential operator is a local operator but the fractional order differential operator is non-local. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. This is more realistic and it is one reason why fractional calculus has become more and more popular [1–11].

In general, there exists no method that yields an exact solution for a fractional differential equation. Only approximate solutions can be obtained using the linearization or perturbation method. The perturbation methods have some limitations e.g., the approximate solution involves series of small parameters which poses difficulty since majority of nonlinear problems have no small parameters at all. Although appropriate choices of small parameters some time leads to ideal solution but in most of the cases unsuitable choices lead to serious effects in the solutions. The homotopy perturbation method (HPM) was first introduced by He [12]. Recently, the homotopy perturbation method has been studied by many authors to handle linear and nonlinear equations arising in physics and engineering [13-17]. The homotopy perturbation transform method (HPTM) is a combination of Laplace transform method, homotopy perturbation method (HPM) and He's polynomials. In recent years, many authors have paid attention to study the solutions of linear and nonlinear partial differential equations by using various methods combined with the Laplace transform. Among these are Laplace decomposition method [18-22] and homotopy perturbation transform method [23-25].

In this paper, we consider the following space fractional telegraph equation of the form:

$$\frac{\partial^\alpha u}{\partial x^\alpha} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + g(t), \qquad t \geq 0,\ 1 < \alpha \leq 2, \tag{1}$$

subject to the initial and boundary conditions:

$$u(0,t) = f_1(t), \qquad t \geq 0, \tag{2}$$

$$\frac{\partial u(0,t)}{\partial x} = f_2(t), \qquad t \geq 0, \tag{3}$$

$$u(x,0) = s(x), \qquad 0 < x < 1, \tag{4}$$

where α is a parameter describing the order of the fractional space-derivative and function $u(x,t)$ is assumed to be a causal function of space, i.e., vanishing for $x < 0$. The derivative is understood in the Caputo sense. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses. In the case of $\alpha = 2$ the fractional telegraph equation reduces to the classical telegraph equation.

Also, we consider the following time fractional telegraph equation

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} + \lambda \frac{\partial^\alpha u}{\partial t^\alpha} = \nu \frac{\partial^2 u}{\partial x^2}, \qquad t \geq 0,\ 0 < \alpha \leq 1, \tag{5}$$

where λ and ν are arbitrary constants and $u(x,t)$ is assumed to be a causal function of time, i.e., vanishing for $t < 0$. In the case of $\alpha = 1$, the fractional telegraph equation reduces to the classical telegraph equation. The fractional telegraph equations have been studied previously by many researchers notably Mamani [26], Yildirim [27] and others.

Further, we apply the new homotopy perturbation transform method (HPTM) to solve the space and time fractional telegraph equations. The objective of the present paper is to extend the application of the HPTM to obtain analytic and approximate solutions to the space and time fractional telegraph equations. The advantage of this technique is its capability of combining two powerful methods for obtaining exact and approximate analytical solutions for nonlinear equations. The fact that the HPTM solves nonlinear problems without using Adomian's polynomials is a clear advantage of this technique over the decomposition method.

II. Basic Definitions of Fractional Calculus

In this section, we mention the following basic definitions of fractional calculus.

Definition1. The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of a function $f(t) \in C_\mu, \mu \geq -1$ is defined as [5]:

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad (\alpha > 0), \tag{6}$$

$$J^0 f(t) = f(t). \tag{7}$$

For theRiemann-Liouville fractional integral we have:

$$J^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\alpha+\gamma}. \tag{8}$$

Definition 2. The fractional derivative of $f(t)$ in the Caputo sense is defined as [8]:

$$D^{\alpha} f(t) = J^{n-\alpha} D^{n} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \tag{9}$$

for $n-1 < \alpha \le n,\ n \in N,\ \ t > 0.$

Definition 3. The Laplace transform of the Caputo derivative is given by Caputo [8]; see also Kilbas et al. [11] in the form

$$L[D^{\alpha} f(t)] = s^{\alpha} L[f(t)] - \sum_{r=0}^{n-1} s^{\alpha-r-1} f^{(r)}(0+),\ (n-1 < \alpha \le n). \tag{10}$$

Definition 4. TheMittag-Lefflerfunctionintroduced by Mittag-Leffler [28]**,** is defined and represented in the following manner:

$$E_{\alpha}(t) = \sum_{k=0}^{\infty} \frac{t^{k}}{\Gamma(\alpha k+1)},\ (\alpha \in C, \mathrm{Re}(\alpha) > 0). \tag{11}$$

III. Basic Idea of theHomotopy PerturbationTransforms Method

To illustrate the basic idea of this method, we consider a general fractional nonlinear nonhomogeneous partial differential equation with the initial conditions of the form:

$$D_t^{\alpha} u(x,t) + R\, u(x,t) + N\, u(x,t) = g(x,t), \tag{12}$$

$$u(x,0) = h(x),\ \ u_t(x,0) = f(x), \tag{13}$$

where $D_t^{\alpha} u(x,t)$ is the Caputo fractional derivative of the function u(x,t), R is the linear differential operator, N represents the general nonlinear differential operator and g(x, t) is the source term. Applying the Laplace transform (denoted in this paper by L) on both sides of Equation (12), we get

$$L[D_t^{\alpha} u(x,t)] + L[R\, u(x,t)] + L[N\, u(x,t)] = L[g(x,t)]. \tag{14}$$

Using the property of the Laplace transform, we have

$$L[u(x,t)] = \frac{h(x)}{s} + \frac{f(x)}{s^2} + \frac{1}{s^{\alpha}} L[g(x,t)] - \frac{1}{s^{\alpha}} L[R\, u(x,t) + N\, u(x,t)]. \tag{15}$$

Operating with the Laplace inverse on both sides of Equation (15) gives

$$u(x,t) = G(x,t) - L^{-1}\left[\frac{1}{s^{\alpha}} L[R\, u(x,t) + N\, u(x,t)]\right], \tag{16}$$

where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now we apply the homotopy perturbation method

$$u(x,t) = \sum_{n=0}^{\infty} p^{n} u_n(x,t) \tag{17}$$

and the nonlinear term can be decomposed as

$$N\,u(x,t)=\sum_{n=0}^{\infty}p^{n}H_{n}(u), \tag{18}$$

for some He's polynomials $H_n(u)$ [29,30] that are given by

$$H_n(u_0,u_1,\ldots,u_n)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty}p^i u_i\right)\right]_{p=0},\ n=0,\,1,\,2,\,3\ldots \tag{19}$$

Substituting (17) and (18) in (16), we get

$$\sum_{n=0}^{\infty}p^n u_n(x,t)=G(x,t)-p\left(L^{-1}\left[\frac{1}{s^\alpha}L\left[R\sum_{n=0}^{\infty}p^n u_n(x,t)+\sum_{n=0}^{\infty}p^n H_n(u)\right]\right]\right), \tag{20}$$

which is the coupling of the Laplace transform and the homotopy perturbation method using He's polynomials. Comparing the coefficients of like powers of p, we have

$$\begin{aligned}
&p^0: u_0(x,t)=G(x,t),\\
&p^1: u_1(x,t)=-L^{-1}\left[\frac{1}{s^\alpha}L\left[R\,u_0(x,t)+H_0(u)\right]\right],\\
&p^2: u_2(x,t)=-L^{-1}\left[\frac{1}{s^\alpha}L\left[R\,u_1(x,t)+H_1(u)\right]\right],\\
&p^3: u_3(x,t)=-L^{-1}\left[\frac{1}{s^\alpha}L\left[R\,u_2(x,t)+H_2(u)\right]\right],
\end{aligned} \tag{21}$$

IV. Applications

In this section, we use the HPTM to solve space and time fractional telegraph equations.

Example 1. Consider the following space fractional telegraph equation

$$\frac{\partial^\alpha u}{\partial x^\alpha}=\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}+u,\qquad t\ge 0,\ 1<\alpha\le 2, \tag{22}$$

subject to the initial and boundary conditions:

$$u(0,t)=e^{-t},\qquad t\ge 0, \tag{23}$$

$$\frac{\partial u(0,t)}{\partial x}=e^{-t},\qquad t\ge 0, \tag{24}$$

$$u(x,0)=e^{x},\qquad 0<x<1. \tag{25}$$

Applying the Laplace transform on the both sides of Equation (22), subject to the initial conditions (23) and (24), we have

$$L[u(x,t)]=\frac{e^{-t}}{s}+\frac{e^{-t}}{s^2}+\frac{1}{s^\alpha}L\left[\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}+u\right]. \tag{26}$$

The inverse Laplace transform implies that

$$u(x,t)=e^{-t}+xe^{-t}+L^{-1}\left[\frac{1}{s^\alpha}L\left[\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}+u\right]\right]. \tag{27}$$

Now applying the homotopy perturbation method, we get

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$$\sum_{n=0}^{\infty} p^n u_n(x,t) = e^{-t} + xe^{-t} + p\left(L^{-1}\left[\frac{1}{s^\alpha}L\left[\frac{\partial^2}{\partial t^2}\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)\right.\right.\right.$$

$$\left.\left.\left.+\frac{\partial}{\partial t}\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)+\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)\right]\right]\right). \quad (28)$$

Comparing the coefficients of like powers of p, we have

$$p^0 : u_0(x,t) = e^{-t}(1+x),$$

$$p^1 : u_1(x,t) = L^{-1}\left[\frac{1}{s^\alpha}L\left[\frac{\partial^2}{\partial t^2}(u_0) + \frac{\partial}{\partial t}(u_0) + u_0\right]\right]$$

$$= e^{-t}\left[\frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\right], \quad (29)$$

$$p^2 : u_2(x,t) = L^{-1}\left[\frac{1}{s^\alpha}L\left[\frac{\partial^2}{\partial t^2}(u_1) + \frac{\partial}{\partial t}(u_1) + u_1\right]\right]$$

$$= e^{-t}\left[\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\right],$$

Therefore, the HPTM series solution is

$$u(x,t) = e^{-t}\left[1 + x + \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)} + \cdots\right]. \quad (30)$$

Setting $\alpha = 2$ in (30), we reproduce the solution of the problem as follows

$$u(x,t) = e^{-t}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right). \quad (31)$$

This solution is equivalent to the exact solution in a closed form

$$u(x,t) = e^{x-t}. \quad (32)$$

It is clear that no linearization or perturbation was used and a closed form solution is obtainable by adding more terms to the HPTM series. The numerical results for the exact solution (32) and the approximate solution (30) obtained by HPTM, for the special case $\alpha = 2$, are shown in Fig. 1. It can be seen from the Fig. 1 that the solution obtained by the present method is nearly identical with the exact solution. The approximate solutions when $\alpha = 1.25$ and $\alpha = 1.75$ are shown by Fig. 2a and 2b respectively. It is to be noted that only the second order term of the HPTM is used in evaluating the approximate solutions for Fig. 2. It is observed that the approximate solution (30) is in full agreement with the results obtained by ADM [26] and HPM [27].

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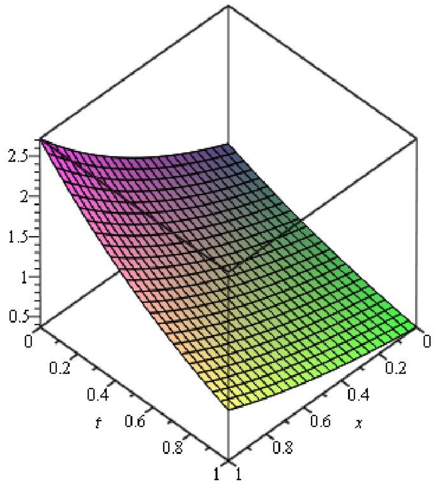

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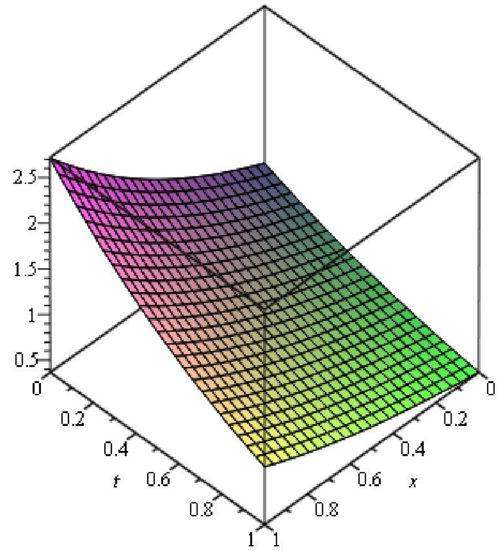

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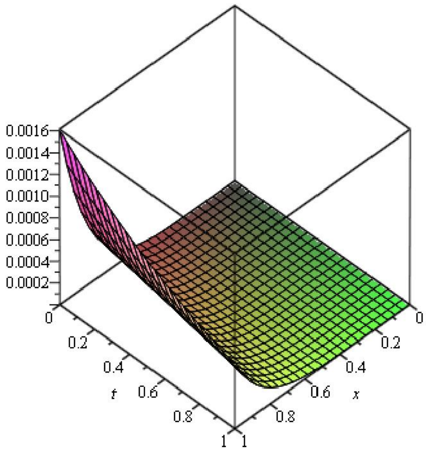
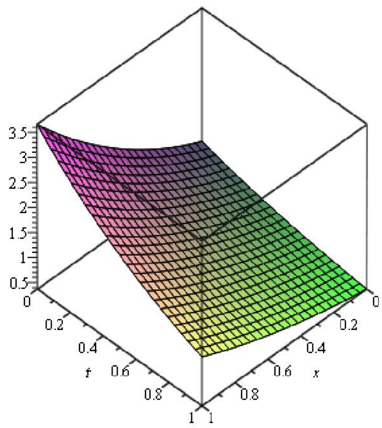
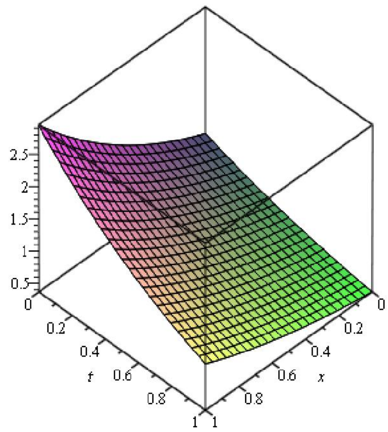


(c)

Fig. 1 The surface shows the solution $u(x,t)$ for equations (22)-(24) when $\alpha = 2$ (a) exact solution (b) approximate solution (c) $|u_{ex} - u_{app}|$.



(a)



(b)

Fig. 2 The surface shows the solution $u(x,t)$ for equations (22)-(24): (a) $\alpha = 1.25$, (b) $\alpha = 1.75$.

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Example 2. Consider the following nonhomogenous space fractional telegraph equation

$$\frac{\partial^\alpha u}{\partial x^\alpha} = \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u - x^2 - t + 1, \qquad t \ge 0,\ 1 < \alpha \le 2, \tag{33}$$

subject to the initial and boundary conditions:

$$u(0,t) = t, \qquad t \ge 0, \tag{34}$$

$$\frac{\partial u(0,t)}{\partial x} = 0, \qquad t \ge 0, \tag{35}$$

$$u(x,0) = x^2, \qquad 0 < x < 1. \tag{36}$$

Applying the Laplace transform on the both sides of Equation (33), subject to the initial conditions (34) and (35), we have

$$L[u(x,t)] = \frac{t}{s} - \frac{2}{s^{\alpha+3}} - \frac{t}{s^{\alpha+1}} + \frac{1}{s^{\alpha+1}} + \frac{1}{s^\alpha} L\left[\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u\right]. \tag{37}$$

The inverse Laplace transform implies that

$$u(x,t) = t - \frac{2x^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{tx^\alpha}{\Gamma(\alpha+1)} + \frac{x^\alpha}{\Gamma(\alpha+1)} + L^{-1}\left[\frac{1}{s^\alpha} L\left[\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u\right]\right]. \tag{38}$$

Now applying the homotopy perturbation method, we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = t - \frac{2x^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{tx^\alpha}{\Gamma(\alpha+1)} + \frac{x^\alpha}{\Gamma(\alpha+1)} + p\left(L^{-1}\left[\frac{1}{s^\alpha} L\left[\frac{\partial^2}{\partial t^2}\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)\right.\right.\right.$$

$$\left.\left.\left. + \frac{\partial}{\partial t}\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right) + \left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)\right]\right]\right). \tag{39}$$

Comparing the coefficients of like powers of p, we have

$$p^0 : u_0(x,t) = t - \frac{2x^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{tx^\alpha}{\Gamma(\alpha+1)} + \frac{x^\alpha}{\Gamma(\alpha+1)},$$

$$p^1 : u_1(x,t) == \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{tx^\alpha}{\Gamma(\alpha+1)} - \frac{2x^{2\alpha+2}}{\Gamma(2\alpha+3)} - \frac{tx^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{40}$$

$$p^2 : u_2(x,t) == \frac{2x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{tx^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{x^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{2x^{3\alpha+2}}{\Gamma(3\alpha+3)} - \frac{tx^{3\alpha}}{\Gamma(3\alpha+1)},$$

Therefore, the HPTM series solution is

$$u(x,t) = \left(t + \frac{2x^\alpha}{\Gamma(\alpha+1)} - \frac{2x^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{2x^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{2x^{2\alpha+2}}{\Gamma(2\alpha+3)}\right.$$

$$\left. - \frac{x^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{2x^{3\alpha+2}}{\Gamma(3\alpha+3)} + \cdots\right). \tag{41}$$

Setting $\alpha = 2$ in (41), we reproduce the solution of the problem as follows

$$u(x,t) = \left(t + \frac{2x^2}{2!} - \frac{2x^4}{4!} + \frac{2x^4}{4!} - \frac{2x^6}{6!} - \frac{x^6}{6!} - \frac{2x^8}{8!} + \cdots\right). \tag{42}$$

We observe that, setting α = 2 in the nth approximations and canceling noise terms yields the exact solution $u(x,t) = x^2 + t$ as $t \to \infty$. The numerical results for the exact solution and the approximate solution (41) obtained by HPTM, for the special case α = 2, are shown in Fig. 3. It can be seen from the Fig. 3 that the solution obtained by the present method is nearly identical with the exact solution. The approximate solutions when α = 1.25 and α = 1.75 are shown by Fig. 4a and 4b respectively. It is to be noted that only the second order term of the HPTM is used in evaluating the approximate solutions for Fig. 4. It is observed that the approximate solution (41) is in full agreement with the results obtained by ADM [26] and HPM [27].

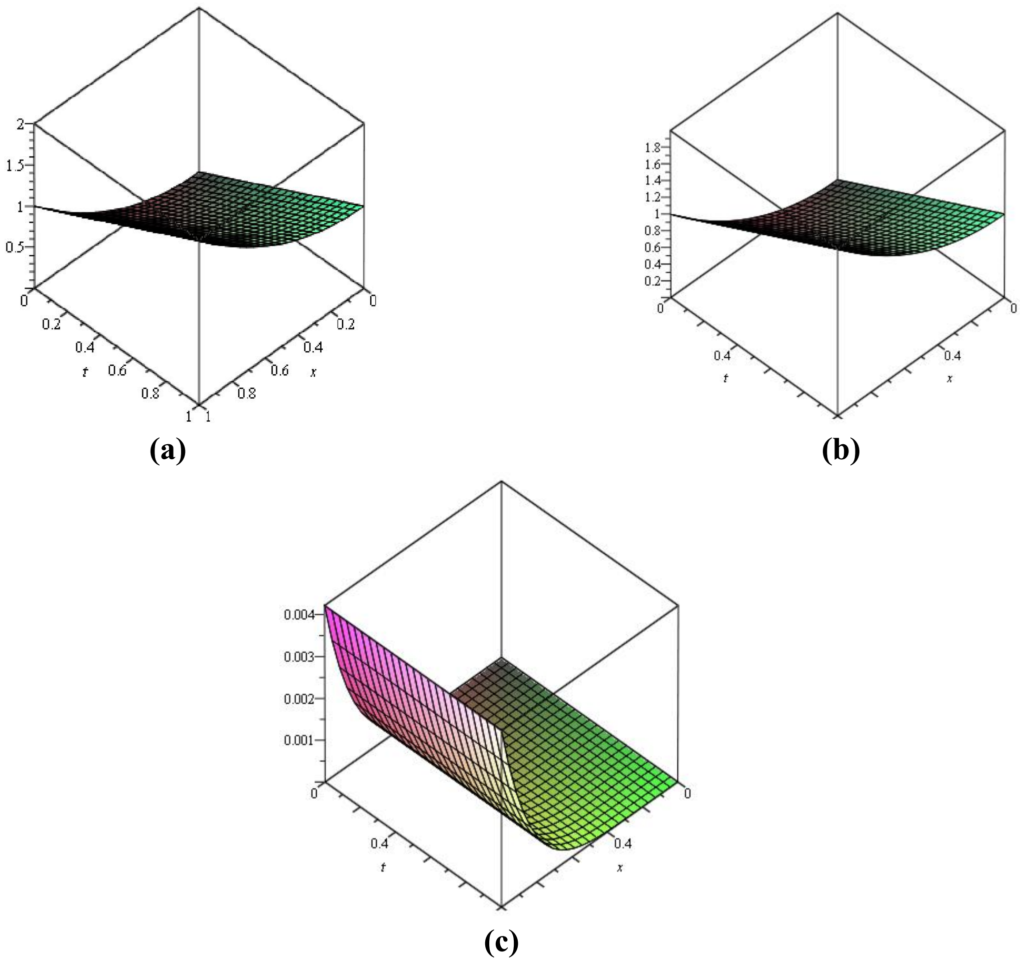

Fig. 3 The surface shows the solution $u(x,t)$ for equations (33)-(35) when α = 2 (a) exact solution (b) approximate solution (c) $|u_{ex} - u_{app}|$.

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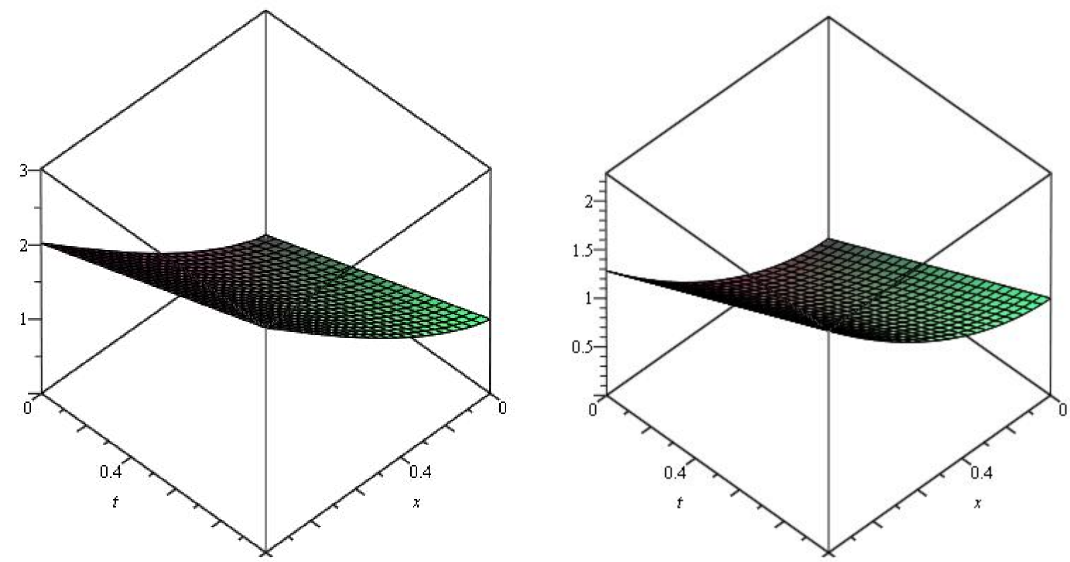

(a) **(b)**

Fig. 4 The surface shows the solution $u(x,t)$ for equations (33)-(35):(a) $\alpha = 1.25$, (b) $\alpha = 1.75$.

Example 3. Consider the following time fractional telegraph equation

$$\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}} + \lambda \frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \nu \frac{\partial^2 u}{\partial x^2}, \qquad t \ge 0,\ 0 < \alpha \le 1, \tag{43}$$

subject to the initial and boundary conditions:

$$u(x,0) = h_1(x), \tag{44}$$

$$u_t(x,0) = h_2(x), \tag{45}$$

$$u(0,t) = s(t). \tag{46}$$

Applying the Laplace transform on the both sides of Equation (43), subject to the initial conditions (44) and (45), we have

$$L[u(x,t)] = \frac{h_1(x)}{s} + \frac{h_2(x)}{s^2} + \frac{1}{s^{2\alpha}} L\left[\nu \frac{\partial^2 u}{\partial x^2} - \lambda \frac{\partial^{\alpha} u}{\partial t^{\alpha}}\right]. \tag{47}$$

The inverse Laplace transform implies that

$$u(x,t) = h_1(x) + t h_2(x) + L^{-1}\left[\frac{1}{s^{2\alpha}} L\left[\nu \frac{\partial^2 u}{\partial x^2} - \lambda \frac{\partial^{\alpha} u}{\partial t^{\alpha}}\right]\right]. \tag{48}$$

Now applying the homotopy perturbation method, we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = h_1(x) + t h_2(x) + p\left(L^{-1}\left[\frac{1}{s^{2\alpha}} L\left[\nu \frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right) - \lambda \frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)\right]\right]\right). \tag{49}$$

Comparing the coefficients of like powers of p, we have

$$p^0 : u_0(x,t) = h_1(x) + t h_2(x),$$

$$p^1: u_1(x,t) = \nu\left[h_1''(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + h_2''(x)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right]$$

$$-\lambda\left[h_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + h_2(x)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right], \qquad (50)$$

$$p^2: u_2(x,t) = \nu^2\left[h_1^{(4)}(x)\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + h_2^{(4)}(x)\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}\right] - 2\nu\lambda\left[h_1''(x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right.$$

$$\left.+ h_2''(x)\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right] + \lambda^2\left[h_1(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + h_2(x)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right],$$

Therefore, the HPTM series solution is

$$u(x,t) = h_1(x) + th_2(x) + \nu\left[h_1''(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + h_2''(x)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right] - \lambda\left[h_1(x)\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right.$$

$$\left.+ h_2(x)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right] + \nu^2\left[h_1^{(4)}(x)\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + h_2^{(4)}(x)\frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}\right]$$

$$-2\nu\lambda\left[h_1''(x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + h_2''(x)\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right] + \lambda^2\left[h_1(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + h_2(x)\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right] + \cdots. \quad (51)$$

Setting $\alpha = 1$ in (51), we reproduce the solution of the problem as follows

$$u(x,t) = h_1(x) + th_2(x) + \nu\left[h_1''(x)\frac{t^2}{2!} + h_2''(x)\frac{t^3}{3!}\right] - \lambda\left[h_1(x)t + +h_2(x)\frac{t^2}{2!}\right]$$

$$+\nu^2\left[h_1^{(4)}(x)\frac{t^4}{4!} + h_2^{(4)}(x)\frac{t^5}{5!}\right] - 2\nu\lambda\left[h_1''(x)\frac{t^3}{3!} + h_2''(x)\frac{t^4}{4!}\right]$$

$$+\lambda^2\left[h_1(x)\frac{t^2}{2!} + h_2(x)\frac{t^3}{3!}\right] + \cdots, \qquad (52)$$

which is the same solution as obtained by using ADM [26] and HPM [27].

V. Conclusion

In this paper, the new homotopy perturbation transform method (HPTM) has been successfully applied for solving space and time fractional telegraph equation. The results obtained by using the HPTM presented here are in full agreement with the results obtained by using ADM [26] and HPM [27]. The method provides the solutions in terms of convergent series with easily computable components in a direct way without using linearization, perturbation or restrictive assumptions. It is worth mentioning that the proposed technique is capable of reducing the volume of the computational work as compared to the classical methods while still maintaining the high accuracy of the numerical result; the size reduction amounts to an improvement of the performance of the approach. Finally, we conclude that the proposed method is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of fractional partial differential equations.

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